1D Kinematics

Motion is relative. Those three small words contain a lot of detail. Essentially, deciding whether something is moving, and describing that motion, require us to use some other reference point for comparison.

Given several different reference points, an object can be not moving at all, moving in some direction at some rate, moving in some other direction at some other rate, and so on – all at the same time. None of those is incorrect; in other words, there is no privileged frame of reference.

Imagine having watched a person walking up and down the hallway. On the floor of the hallway, someone has put down a strip of tape with position markers on it (similar to the yard line markers on a football field). Because you are the curious sort, you started a timer and made some notes about where the person was located at various times. Here is your raw data:

When you pressed the "start" button on the timer, the person was at the 3 meter mark already moving.

At time of 1 second: person at 5 meter mark

time = 2 s : position = 7 meters   time = 3 s : position = 9 m

time = 4 s : position = 11 m   time = 5 s : position = 13 m

time = 6 s : position = 13 m   time = 7 s : position = 9 m

We'll use the symbol "t" for time, and since the person is moving along a line in the hallway, we'll use the symbol "x" for position.

Let's put that raw data into a data table:

t (s)	X (m)
0 s	3 m
1 s	5 m
2 s	7 m
3 s	9 m
4 s	11 m
5 s	13 m
6 s	13 m
7 s	9 m

Remember: $X \equiv$ position , $t \equiv$ time

★ t is a point in time, i.e. a place on the timeline; it is NOT necessarily how much time has elapsed. In other words, t is NOT a time interval — it is a point in time.

★ Notice that the unit of measure (e.g. seconds, meters) is explicitly stated in each box.

The data table is certainly more useful to us than the raw data; I suspect you can already see some patterns. However, a graph would be even better.

To make a graph, we need to determine which variable is the independent variable, which is the dependent variable, and what scale to use on our axes.

Since you recorded the person's position at various times, it would be appropriate to say that the person's position was dependent on time. Therefore:

- independent variable → time (t)
- dependent variable ⟶ position (x)

★ independent variable is sometimes called "manipulated variable" and dependent variable is sometimes called "responding variable"

The independent variable should be graphed on the horizontal axis...

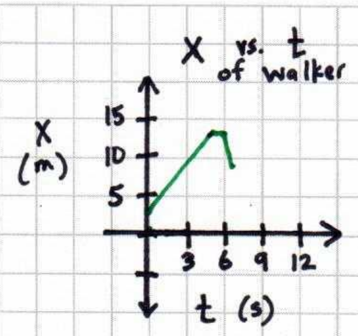
★ the horizontal axis is usually called the "x axis", but that's just asking for confusion here, right? :)

The dependent variable should be graphed on the vertical axis (usually called the "y axis")

To determine the needed scale of the axes, look at the range of values that you need to graph on each axis.

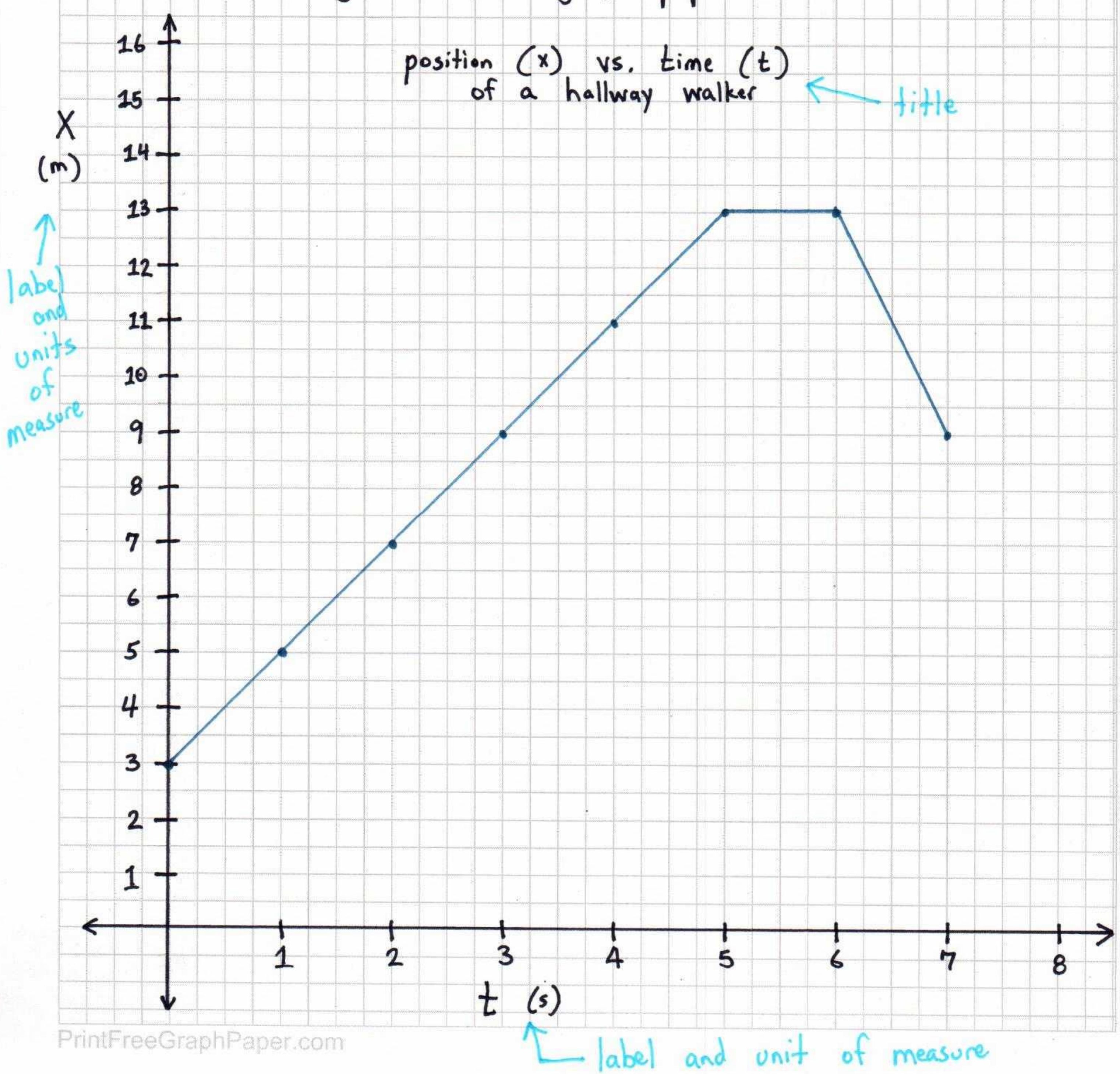
For time, the values range from 0 seconds to 7 seconds.
For position, they range from 3 meters to 13 meters.

We should be safe to go from 0 to 10 seconds on the time axis and from 0 to 15 meters on the position axis.



This graph is titled, scaled appropriately, and both axes are labeled and have correct units of measure. It's not very useful though, because it's too small.

We have a big sheet of graph paper – let's use more of it.



We can get a lot of information from that graph.

Suppose I asked what distance the person traveled during the time you were collecting data.

Distance is the total length of the path traveled by an object, and we'll use "d" to represent that.

The person started at the 3 m mark and moved to then 13 m mark, then moved back to the 9 m mark.

Moving from 3 m to 13 m was $(13\,m - 3\,m) = 10\,m$ of distance, and then moving from 13 m to 9 m was $(9\,m - 13\,m) = -4\,m$ but distance is a length and can't be negative so it's just 4 m. Put those two together for a total distance of $\boxed{d = 14\ m}$

Distance is known as a scalar quantity because it does not have a directional component; in other words, it doesn't matter if you travel left, right, east, north, up, or down – if you walked a path that was 14 m long, then your distance traveled was 14 m.

★ scalar quantity – has no directional component; there is only a magnitude (an amount)

Since we now know the person's distance traveled, we can determine how quickly that distance was traveled. We call that "speed" – the rate at which distance is traveled – and use the letter "v" to represent it.

* "s" seems like a better choice, but lots of math classes use "s" for position, so I don't want to condition you into associating "s" with "speed".

Knowing that speed is a rate, and recalling from prior math classes that a rate implies a function of time (* I have to throw in a "not necessarily" here, but for now, that's good enough), then it seems as if speed would be distance divided by time, or $v = d/t$ In fact, you probably already knew that. Well, it's wrong.

To explain, consider this: on January 4^{th}, it is day 4 of the year. When you walk 12 feet from your desk to the pencil sharpener on January 4^{th}, surely you are not suggesting that your speed is this:

$$d = 12 \text{ ft}, \quad t = 4 \text{ days} \longrightarrow v = \frac{d}{t} = \frac{12 \text{ ft}}{4 \text{ days}} = 3 \text{ ft/day}$$

If so, I'd argue that you need a more active life. :(

Instead, it's hopefully obvious that we need to use the amount of time it took to travel that 12 ft, right?

What we actually need is the time interval over which our hallway walker traveled that 14 m of distance. We express intervals as a difference between the ending value and the starting value. In this case, it's the difference between the time you stopped recording ($t = 7\,s$) and the time you started recording ($t = 0\,s$).

Writing out "the difference in t" or "the change in t" is going to be annoying though, so we'll use a letter from the Greek alphabet to represent "the change in" stuff. Greek letter "delta" ~~(lowercase)~~ (uppercase) looks like this: Δ

$\Delta \equiv$ the change in / the difference between

This makes our equation for speed look like this:

$$V = \frac{d}{\Delta t}$$

How to calculate Δt? It's the ending t minus the starting t, which looks like this: $t_f - t_i$

Note the subscripts "f" and "i" which mean, respectively, "final" and "initial"

★ This idea applies any time you see delta something:

$$\Delta t = t_f - t_i \qquad \Delta r = r_f - r_i$$

$$\Delta \ell = \ell_f - \ell_i \qquad \Delta A = A_f - A_i$$

We can finally calculate the speed of our hallway walker.

$$V = \frac{d}{\Delta t} \qquad d = 14\text{ m} \qquad t_f = 7\text{s}$$
$$\Delta t = t_f - t_i \qquad t_i = 0\text{s}$$

$$V = \frac{14\text{ m}}{7\text{s} - 0\text{s}} = \frac{14\text{m}}{7\text{s}} = 2\text{ m/s}$$ ← Notice that the unit of measure is included...

Was our hallway walker traveling at 2 m/s the entire time?

Hint: Look at the interval between 5 s and 6 s ...

At $t = 5$ s, the person was on the 13 m mark.
At $t = 6$ s, the person was on the 13 m mark.
For the entire time ~~internal~~ interval between 5 s and 6 s inclusive, the person was located on the 13 m mark. It certainly seems like the person was not moving at all during that time. What gives?

What we calculated above (2 m/s) is the walker's average speed over the total time in which data was recorded. The walker's speed at any given time is not required to be the same as the average speed. That should be sensible to you – after all, your average speed from one city to another may be 60 mi/hr, but I dare say you didn't go through the stop signs and red lights at 60 mi/hr, right? Right? ...

So we need to clarify our equation for speed: in words, it's something along these lines:

The average speed of an object is its total distance traveled divided by the amount of time required to travel that distance.

In mathematical terms: $V_{avg} = \frac{d_{total}}{\Delta t_{total}}$

Determing the person's speed at a particular instant in time is a bit more involved, but it will be easier to understand if we do something else first...

We established earlier that distance is a scalar quantity (see page 5). Since distance does not include a directional component, it would be reasonable to conclude that any other quantity based on distance is also scalar. In fact, multiplying or dividing scalar quantities by other scalar quantities will always result in a scalar quantity. Therefore, speed is a scalar quantity.

As interesting and sometimes useful distance and speed are, it would be good to consider the direction(s) that our hallway walker moved as he/she traveled.

As stated earlier, distance as a concept is certainly useful to us. We like to know the odometer reading on used cars because the total distance it's traveled over its lifetime is useful information. If you run for exercise, you want to keep up with your total distance traveled.

However, it would also be useful to compare where the person started and where they ended up. In other words, what was the person's change in position during the data collection? To find this, we need to know the person's position at the beginning (initial position) and at the end (final position). * see page 4

At $t = 0\,s$, the person was at the 3 m mark ($x_i = 3\,m$)

At $t = 7\,s$, the person was at the 9 m mark ($x_f = 9\,m$)

Change in position $\equiv \Delta\vec{x} = x_f - x_i = {}^{+}9m - {}^{+}3m = {}^{+}6\,m$

That tells us that the person's position changed by 6 m in the positive direction from 0 s to 7 s.

Notice that this quantity has a directional component – it is what we call a vector quantity.

* Vector quantity – has a directional component in addition to an amount; direction is important an is an integral part of the quantity

You may have noticed something different about the way I wrote the variable for change in position – it has a half-arrow above the letter ($\Delta\vec{x}$). This is a nice way to let you know that it is a vector.

There is a (somewhat) shorter name for "change in position" – it's called "displacement"

★ change in position = displacement $\equiv \Delta\vec{x} = x_f - x_i$

In the example already given, we calculated $\Delta\vec{x}$ for the entire 7 s the person was walking. However, we will often be interested in only a portion of the trip.

Let's look a the time interval from zero to five seconds:

@ $t = 0\,s$, $x = 3m$; @ $t = 5\,s$, $x = 13\,m$

$\Delta\vec{x}$ for $t = 0s$ to $t = 5s$ then is:

$$\Delta\vec{x} = x_f - x_i = {}^{+}13\,m - {}^{+}3\,m = {}^{+}10\,m$$

How about from 5 to 6 seconds?

@ $t = 5s$, $x = 13\,m$; @ $t = 6\,s$, $x = 13\,m$

$$\Delta\vec{x} = x_f - x_i = {}^{+}13m - {}^{+}13m = 0\,m$$

From 6 to 7 s ?

@ $t = 6\,s$, $x = 13\,m$; @ $t = 7\,s$, $x = 9\,m$

$$\Delta\vec{x} = x_f - x_i = {}^{+}9m - {}^{+}13m = {}^{-}4\,m$$

Let's recap what we found with respect to displacements:

Δt from 0 s to 7 s : $\Delta\vec{x} = {}^{+}6\,m$

Δt of 0 to 5 s : $\Delta\vec{x} = {}^{+}10\,m$

Δt of 5 to 6 s : $\Delta\vec{x} = 0\,m$ — Notice anything here? :

Δt of 6 to 7 s : $\Delta\vec{x} = {}^{-}4\,m$

In case you didn't notice, the combined $\Delta\vec{x}$ for each of the smaller time periods ($^{+}10\,m + 0\,m + {}^{-}4\,m$) is exactly the same as the $\Delta\vec{x}$ for the entire time period ($^{+}6\,m$).

At the very least, this indicates that vectors can be added together to get another vector.

At this point, we can tell that distance and displacement are similar, but they are certainly not identical.

distance (d) is scalar (no directional component) and tells us the length of the path traveled by an object.

displacement ($\Delta\vec{x}$) is a vector (has a directional component) and tells us the objects' change in position.

Since distance and displacement are similar, and speed is the rate at which distance is traveled, it seems reasonable to expect that there is a vector quantity related to displacement...

Displacement ($\vec{\Delta x}$) is, as we said earlier, an object's change in position, and since it is a vector, the direction of that change is included.

The rate of change in position is called velocity ($\vec{v}$). Note that it uses the same letter as speed (v), but velocity ($\vec{v}$) has the half-arrow above it. This is a handy way to distinguish between scalar and vector quanties.

Earlier we found that our hallway walker's total displacement over the entire seven seconds was +6 m. We can easily find that person's average velocity for that time interval:

$$\vec{V}_{avg} = \frac{\Delta \vec{x}}{\Delta t} = \frac{+6\,m}{7\,s} \cong +.86\ m/s$$

* In vector quantities, the sign (+/-) is a nice convenient way to indicate direction. In this specific case, it can only mean which direction with respect to "more positive numbers" or "more negative numbers."

However, in a different scenario, +/- could mean north / south, up/down, left/right, and so on...
There is no particular reason that e.g. "up" has to be the "+" direction, so don't create any bad habits...

As hinted earlier, there's no good reason for average velocity to be indicative of velocity at some particular instant in time – after all, an object can slow down, speed up, change direction, or even temporarily come to a stop during a trip. That being the case, we need some way to determine the velocity of an object at a particular instant in time.

Look back at the graph on page 4 – notice that there are three discrete sections of the graph:

- from zero to five seconds
- from five to six seconds
- from six to seven seconds

Let's analyze zero to five seconds first.

At $t = 0\,s$, $x = {}^{+}3\,m$; At $t = 5\,s$, $x = {}^{+}13\,m$

$$\Delta\vec{x} = x_f - x_i = {}^{+}13m - {}^{+}3m = {}^{+}10m$$

$$\vec{V}_{avg} = \frac{\Delta\vec{x}}{\Delta t} = \frac{{}^{+}10m}{5s - 0s} = \frac{{}^{+}10m}{5s} = {}^{+}2\,m/s$$

So average velocity from 0 to 5_s is ${}^{+}2\,m/s$.

That section of the graph is linear, which tells us that the ratio of vertical change ($\Delta\vec{x}$) compared to horizontal change (Δt) is constant – in other words, the slope is constant, which means it's linear, which is exactly what was already stated...

* I'd make a pun about this being circular, but that might confuse the issue... :)

Since the zero to five second interval of the graph is linear, we can do something with it: let's find the slope of that section.

$$slope = \frac{rise}{run}$$

Since "rise" is change in vertical axis, and "x" is on our vertical axis: $rise = \Delta\vec{x}$

Since "run" is change in horizontal axis, and "t" is on our horizontal axis: $run = \Delta t$

Therefore, slope for this graph is:

$$slope = \frac{rise}{run} = \frac{\Delta\vec{x}}{\Delta t}$$

Interestingly enough, that looks exactly like our equation for average velocity: $\vec{v}_{avg} = \frac{\Delta\vec{x}}{\Delta t}$

The fact that the graph is linear during the interval from zero to five seconds means that the rate of change in x was constant over that time period. In essence, the velocity (rate of change in x) at any given point during that interval is the same as the average velocity over the entire interval.

Moral of the story: instantaneous velocity ($\vec{v}_{inst}$) is the slope of the x vs t graph at some time t.

Let's elaborate on that a bit:

$\vec{V}_{inst}$ = slope of x vs t graph at some time t

This idea works regardless of whether the graph of x vs t is linear. It's beyond the scope of these (introductory) notes, but if you "zoom in" far enough on a curve, it appears to be linear, and the time interval you'll be looking at will be really small. The "slope" of that "linear" segment will be the instantaneous velocity at that specific time t. Mathematically, it looks like this: $\vec{V}_{inst} = \lim_{\Delta t \to 0} \frac{\Delta \vec{x}}{\Delta t}$ In words, it's along the lines of "instantaneous velocity is the rate of change in x over a time interval that is really really close to zero but not quite zero."

In calculus, you'll refer to that as the derivative of x with respect to time, or dx/dt

To summarize:

$$\vec{V}_{inst} = \lim_{\Delta t \to \emptyset} \frac{\Delta \vec{x}}{\Delta t} = \frac{dx}{dt} = \text{slope of } x \text{ vs } t \text{ at time } t$$

Before we move on, let's make sure this works for the other two sections of the graph on page 4...

From five to six seconds, our hallway walker appears to have had no change in position:

@ $t = 5s$: $x = {}^{+}13m$

@ $t = 6s$: $x = {}^{+}13m$

That being the case, we should expect it to have a velocity of zero during that time interval.

* $\vec{V}_{inst}$ = slope of x vs. t at time t

Since x vs. t is linear from $t = 5s$ to $t = 6s$, we know that the average velocity during that interval is the same as the instantaneous velocity at all points in that interval.

* slope from $t = 5s$ to $t = 6s$ $= \emptyset$

So that checks out as expected.

Now let's check the last section ($t = 6s$ to $t = 7s$).

Same rationale as above (it's linear there), so:

$\vec{V}_{inst}$ = slope of x vs t at t

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta \vec{x}}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$$

@ $t = 6s$: $x = {}^{+}13m$

@ $t = 7s$: $x = {}^{+}9m$

$$= \frac{{}^{+}9m - {}^{+}13m}{7s - 6s} = \frac{-4m}{1s} = -4\ m/s$$

We'll recap all of this on the next page...

Let's recap what we've found so far:

From $t = 0\,s$ to $t = 5\,s$: $\Delta\vec{x} = {}^{+}10\,m$, $\vec{V}_{avg} = {}^{+}2\,m/s$

From $t = 5\,s$ to $t = 6\,s$: $\Delta\vec{x} = 0\,m$, $\vec{V}_{avg} = 0\,m/s$

From $t = 6\,s$ to $t = 7\,s$: $\Delta\vec{x} = -4\,m$, $\vec{V}_{avg} = -4\,m/s$

Since $\vec{V}_{avg} = \frac{\Delta\vec{x}}{\Delta t}$, a bit of algebraic manipulation means $\Delta\vec{x} = \vec{V}_{avg} \cdot \Delta t$

We can use that to verify our findings above.

From $t = 0\,s$ to $t = 5\,s$:

$$\Delta\vec{x} = \vec{V}_{avg} \cdot \Delta t = {}^{+}2\,m/s \cdot 5\,s = {}^{+}10\,m \checkmark$$

From $t = 5\,s$ to $t = 6\,s$:

$$\Delta\vec{x} = \vec{V}_{avg} \cdot \Delta t = 0\,m/s \cdot 1\,s = 0\,m \checkmark$$

From $t = 6\,s$ to $t = 7\,s$:

$$\Delta\vec{x} = \vec{V}_{avg} \cdot \Delta t = -4\,m/s \cdot 1\,s = -4\,m \checkmark$$

As we did before, we can take the vector sum of these displacements to find the total displacement for the total time interval:

$$\Delta\vec{X}_{total} = {}^{+}10\,m + 0\,m + {}^{-}4\,m = {}^{+}6\,m \checkmark$$

This checks out with our earlier result...

Note that we can NOT do the same with the average velocities for each interval. You can't add them up to get anything at all meaningful. You can't average them to get anything meaningful. * You could use a weighted average, but we aren't going there...

Our hallway walker was nice enough to travel at constant velocities during each of the discrete sections of the graph on page 4, but the "Real World" isn't often so kind to us. In fact, the graph we had was overly simplified. According to that graph, the walker's velocity changed from +2 m/s to ∅ m/s at $t = 5$ s. We don't see it on the graph, but it MUST have taken some time for that change in velocity to occur – it couldn't have occurred instantaneously...

* Note that the velocity change happened AT 5 s, NOT over a time interval of 5 s...

Since we can't use our other graph to study how velocity changes over time, we'll have to come up with something else. Hopefully you've seen a "fan cart" by now – if not, it's essentially a small car with a large fan mounted on a swivel on top of the car.

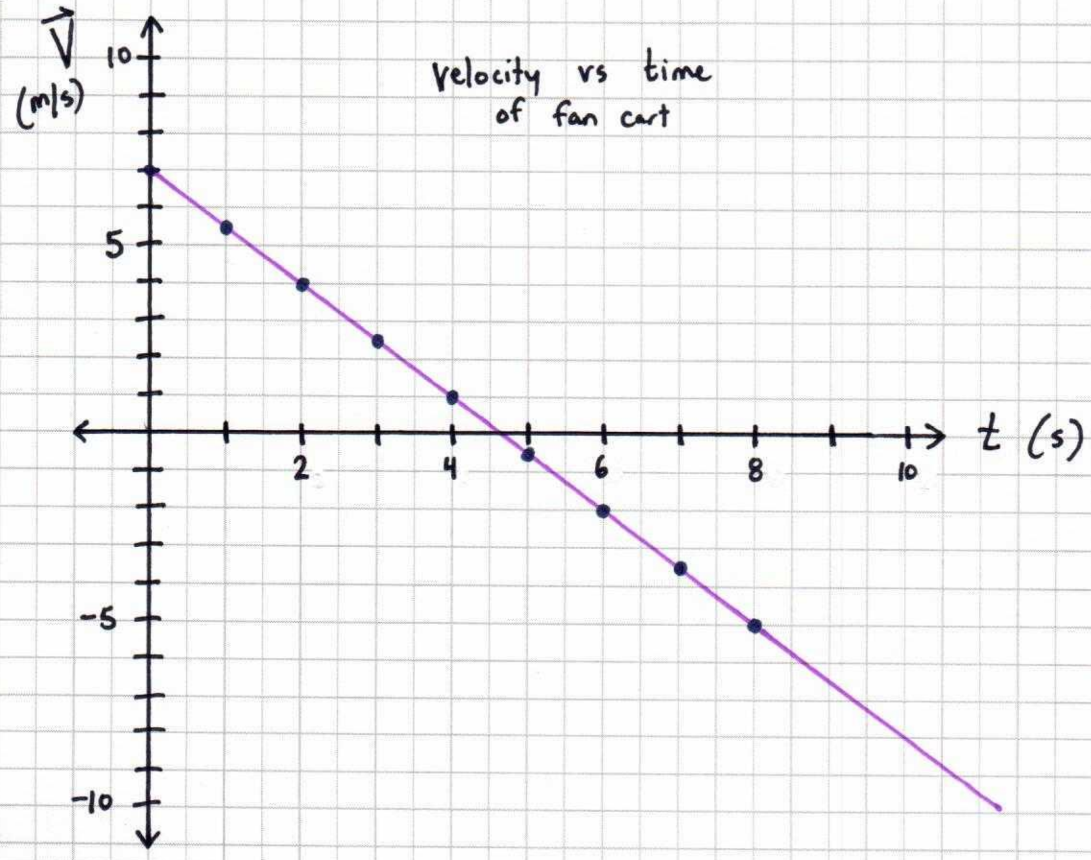
We're going to turn the fan cart on, give it a push in some direction across the floor, and use some magic* to record its velocity at various times.

* it's not really magic, of course; we can easily do this with a motion sensor

Here's the data from our motion sensor, nicely placed into a data table:

time, t	Ø s	1 s	2 s	3 s	4 s	5 s	6 s	7 s	8 s
velocity, $\vec{v}$	+7 m/s	+5.5 m/s	+4 m/s	+2.5 m/s	+1 m/s	-.5 m/s	-2 m/s	-3.5 m/s	-5 m/s

Note that we have a starting velocity that's positive (+). We said earlier that + and − with respect to vectors indicate direction of travel, and since we pushed the cart across the floor, it seems safe to conclude that the positive direction is away from us. Taking into account the later negative velocities, let's make a graph:



The first thing you might notice is that this graph of $\vec{v}$ vs. t is linear, which is to say that the rate of change of $\vec{v}$ over time is constant. We can find that rate using the slope of the graph:

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

For this graph of $\vec{v}$ vs. t, rise $= \Delta\vec{v}$ and run $= \Delta t$

$$\text{slope} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i} = \frac{-5\text{ m/s} - {}^{+}7\text{ m/s}}{8\text{ s} - 0\text{ s}} = \frac{-12\text{ m/s}}{8\text{ s}} =$$

$$= -\frac{3}{2}\,\frac{\text{m/s}}{\text{s}} = -1.5\,\frac{\text{m/s}}{\text{s}}$$

So the average rate of change in the car's velocity is $-1.5\,\frac{\text{m/s}}{\text{s}}$, or in words, -1.5 m/s every s, or negative 1.5 m/s during every second.

Since this graph is linear, we don't need to worry too much about the distinction between the average rate of change in velocity versus the instantaneous rate of change in velocity, but we certainly could – it works the same way that it did with position:

* instantaneous rate of change in velocity =

$$\lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{dv}{dt}$$

↑ calculus

We have a term for "rate of change in velocity" — it's called "acceleration." That simplifies things a bit:

$$\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t} \quad \text{OR} \quad \vec{a}_{avg} = \frac{\vec{v}_f - \vec{v}_i}{\Delta t}$$

$$\vec{a}_{inst} = \lim_{\Delta t \to \emptyset} \frac{\Delta\vec{v}}{\Delta t} = \frac{dv}{dt} = \text{slope of } \vec{v} \text{ vs. } t \text{ at time } t$$

Since velocity is a vector (it has a directional component), it seems reasonable for any change to velocity to also have a direction — a positive velocity can change to be more positive or less positive, for example. Therefore, acceleration must also be a vector quantity.

We found our cart to have an acceleration of $\vec{a} = -1.5 \frac{m/s}{s}$ This is a very intuitive way of writing it (and saying it: "negative 1.5 meters per second per second" says exactly what it means), but you'll often see it written differently:

$$\frac{m/s}{s} = \frac{m}{s} \div s = \frac{m}{s} \cdot \frac{1}{s} = \frac{m}{s^2}$$

That means you'll usually see the acceleration of our fan cart given as $\vec{a} = -1.5\ m/s^2$

★ Be able to manipulate units of measure as I did above; this will be a VERY handy skill for you...

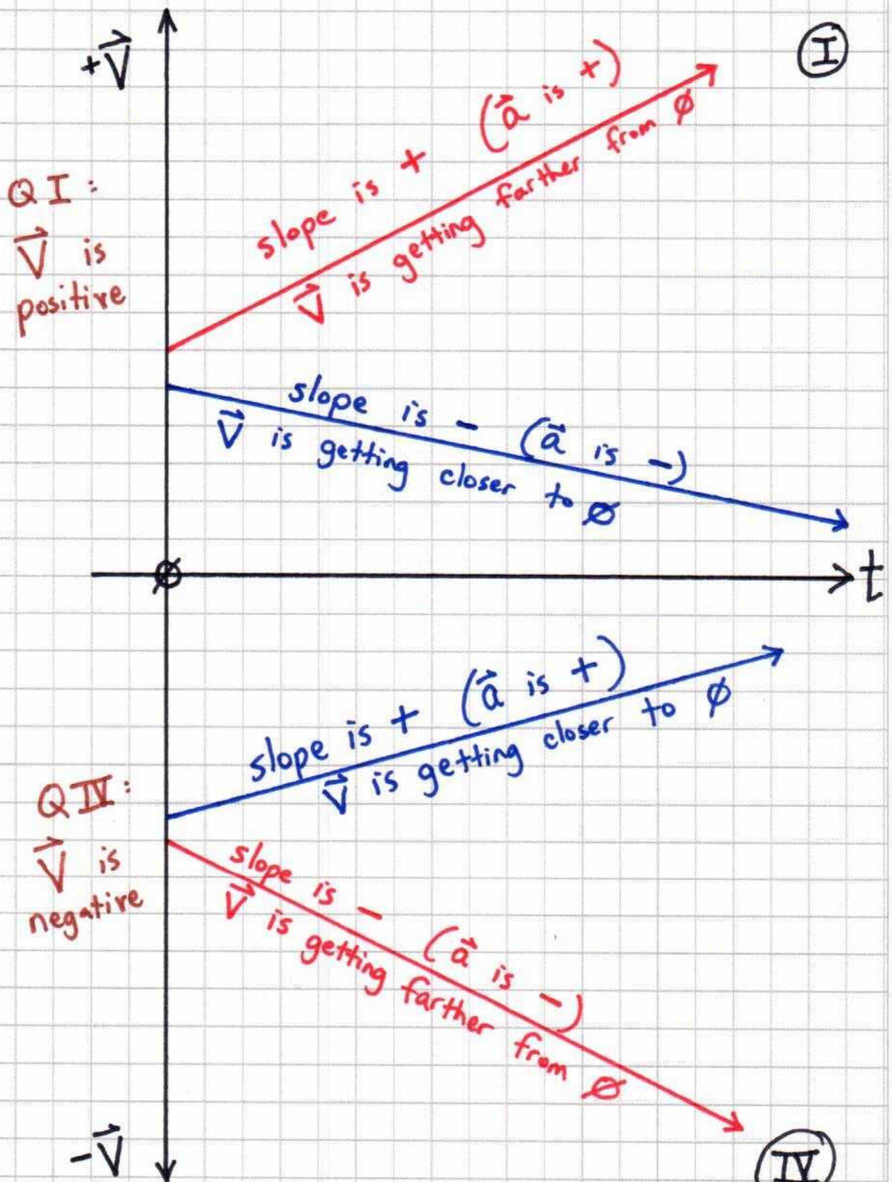
Let's make sure we understand the full meaning of acceleration – consider quadrants I and IV of a standard coordinate plane:

If $\vec{V}$ is positive and $\vec{a}$ is positive, then $\vec{V}$ is getting farther from zero (speed is increasing)

If $\vec{V}$ is positive, and $\vec{a}$ is negative, then $\vec{V}$ is getting closer to zero (speed is decreasing)

If $\vec{V}$ is negative, and $\vec{a}$ is positive, then $\vec{V}$ is getting closer to zero (speed is decreasing)

If $\vec{V}$ is negative, and $\vec{a}$ is negative, then $\vec{V}$ is getting farther from zero (speed is increasing)



In short, if an object is accelerating in the same direction as its velocity, it is increasing in speed. If the object's velocity and acceleration are in opposite directions, then it is decreasing in speed.

Acceleration doesn't necessarily mean that speed has changed though. Given that acceleration is the rate of change in velocity, and given that velocity is a vector (it has direction embedded in it), it seems reasonable for any change in velocity (even if only a change in direction) to qualify as acceleration. In other words, an object can accelerate by changing direction even if it maintains a constant speed while doing so...

A good example of this is an object being swung around in a circular pattern while attached to a string. Even though the object can be kept at a constant speed, it is continually changing direction to stay in a circular pattern. This is known as Uniform Circular Motion (UCM), which will be covered at a later date...

Let's go back to our velocity vs. time graph for the moment... We noted earlier that the graph was linear, meaning the rate of change in velocity (acceleration) was constant over that time interval. We like lines, right?
We know how to do stuff with lines – stuff like finding slope, y-intercept, midpoint, etcetera...

On our $\vec{V}$ vs. t graph, we have a nice line segment between $(t = 0\,s,\ \vec{V} = {}^{+}7\ m/s)$ and $(t = 8\,s,\ \vec{V} = {}^{-}5\ m/s)$. In math classes, you'd probably write those as $(0, 7)$ and $(8, {}^{-}5)$. Since velocity is changing at a constant rate (it's linear, right?), then the average velocity should be the middle velocity. That should be easy enough – we'll just use that thing you learned in math class as "the midpoint formula" or something along those lines:

$$\text{Midpoint} = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$$

Slightly modified for our purposes, it looks like this:

$$\text{Midpoint} = \left(\frac{t_2 + t_1}{2}, \frac{\vec{V}_2 + \vec{V}_1}{2}\right)$$

Applying that with proper substitution gives us:

$$\text{Midpoint} = \left(\frac{8 + 0\ s}{2}, \frac{-5 + 7\ m/s}{2}\right) = \left(4\,s, {}^{+}1\ m/s\right)$$

We don't much care about the time part here, aside from making sure that $t = 4\,s$ is indeed halfway between $t = 0\,s$ and $t = 8\,s$, but knowing that the middle velocity is ${}^{+}1$ m/s is certainly useful. In fact, we can now conclude this:

$$\vec{V}_{avg} = \frac{\vec{V}_i + \vec{V}_f}{2} \quad \text{IF } \vec{a} \text{ is constant}$$

So now we know that we can determine an object's acceleration from its velocity vs. time graph, and if that graph is linear (it will be for this course), we can also determine its average velocity.

But WAIT! THERE'S MOAR!

Let's start simple: consider an object moving at a velocity of +2 m/s for a time period of 5 s. We have some equation stuff we could use:

$$\vec{V}_{avg} = \frac{\Delta \vec{X}}{\Delta t} \longrightarrow \Delta \vec{X} = \vec{V}_{avg} \cdot \Delta t$$

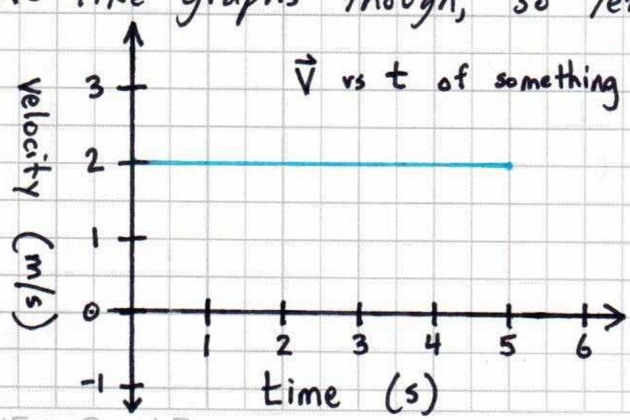
Since $\vec{V}$ is constant over the 5 s time period, we know that $\vec{V}_{avg}$ is $\vec{V}$ the entire time. That plus the equation above leads to this:

$$\Delta \vec{X} = \vec{V} \cdot \Delta t = {}^{+}2 \, \mathrm{m/s} \cdot 5 \, \mathrm{s} = {}^{+}10 \, \mathrm{m}$$

★ Check your units of measure:

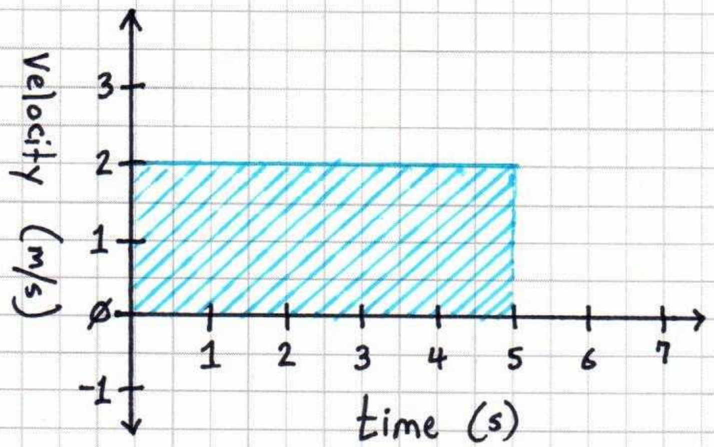
$$\frac{m}{s} \cdot s = \frac{m \cdot s}{s} = m$$

We like graphs though, so let's put that on a graph:



Notice anything???

Same graph – look again...



See that rectangle now? What is its area?

The base of that rectangle is 5 s. The height of that rectangle is $^{+}2$ m/s. Like all rectangles, its area is its base times height: $5\text{ s} \cdot {}^{+}2\text{ m/s} = {}^{+}10\text{ m}$

Isn't that interesting?

As it turns out, the area between the velocity "curve" and time axis on a velocity vs. time graph represents the displacement of the object during that time interval. If the shaded area is above the time axis ($+\vec{v}$), it represents positive displacement, while area below the time axis (between it and the velocity "curve", $-\vec{v}$) represents negative displacement. Whether the area is above or below the time axis, it is referred to as "area under the curve."

$\Delta\vec{x}$ = area under velocity vs. time curve

This "area under the curve" thing is nice, but as we've already seen, the "Real World" isn't so nice about keeping velocity constant for us. Let's consider that same object starting out at $\vec{v} = {}^{+}2\ m/s$ but with some constant acceleration:

The object starts at $^{+}2$ m/s and accelerates to $+9.5$ m/s over a 5 s time period

We can easily calculate the acceleration: $\vec{a} = \frac{\Delta \vec{v}}{\Delta t}$

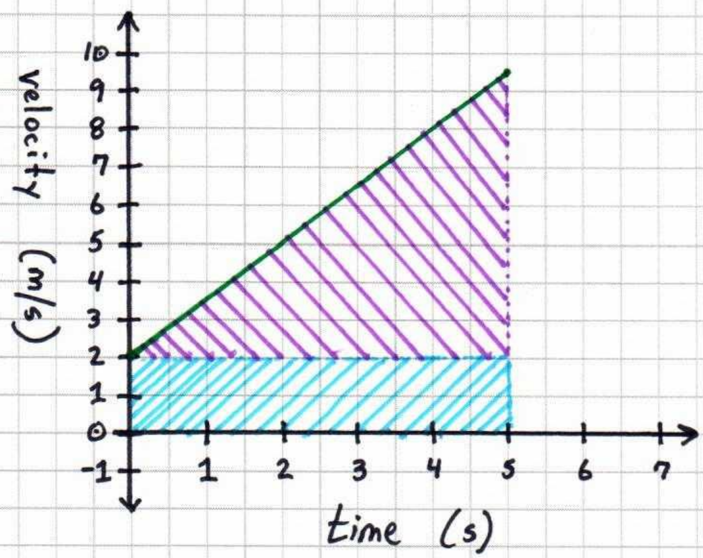
$$\vec{a} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i} = \frac{+9.5\ m/s - {}^{+}2\ m/s}{5\ s\ -\ \emptyset\ s}$$

$$\vec{a} = \frac{+7.5\ m/s}{5s} = {}^{+}1.5\ m/s^2$$

Calculating the acceleration is nice and perhaps desired, but in this case, we'd like to know the displacement of the object. That should be the area under the curve from $t = \emptyset$ s to $t = 5$ s, but this isn't a nice rectangle like the previous example... Hmm...

That looks maybe like a trapezoid, or a triangle stacked on top of a rectangle... Well, we know how to find the area of a rectangle, and a triangle is just half of a rectangle, so...

Let's sketch that out again:



The rectangle part is easy (we already did that earlier):

$$A_{\square} = b \cdot h = 5s \cdot {}^{+}2\ m/s = {}^{+}10\ m$$

Note that this is the initial velocity times the time interval $(\vec{V}_i \cdot \Delta t)$

The triangle is simple too:

$$A_{\Delta} = \frac{1}{2} \cdot b \cdot h = \frac{1}{2} \cdot 5s \cdot ({}^{+}9.5\ m/s - {}^{+}2\ m/s)$$

$$= \frac{1}{2} \cdot 5s \cdot {}^{+}7.5\ m/s = {}^{+}18.75\ m$$

* Notice that we used the height of the triangle itself (we had to account for the height of the rectangle underneath)

Note that $\frac{1}{2}bh$ is actually $\frac{1}{2} \cdot \Delta t \cdot (\vec{V}_f - \vec{V}_i)$

We now have the displacement the object would have had if its velocity were constant — the rectangle part: $^{+}10\ m$ and the displacement due to its acceleration — the triangle part: $^{+}18.75\ m$ Putting them together, we get this: $^{+}10\ m + {}^{+}18.75\ m = {}^{+}28.75\ m$

That's nice and all, but we really need a way to check this and see if it's reasonably correct...

We had already established that $\vec{V}_{avg} = \frac{\Delta \vec{X}}{\Delta t}$ earlier, and back on page 25, we determined that, in cases where acceleration is constant, $\vec{V}_{avg} = \frac{\vec{V}_i + \vec{V}_f}{2}$.

Since the slope of our $\vec{V}$ vs t graph is constant, and since slope of $\vec{V}$ vs t is acceleration, we can safely conclude that $\vec{a}$ is constant here. Therefore, let's do some checking:

$$\vec{V}_{avg} = \frac{\Delta \vec{X}}{\Delta t} \quad \text{AND} \quad \vec{V}_{avg} = \frac{\vec{V}_i + \vec{V}_f}{2} \quad \text{so:}$$

$$\frac{\Delta \vec{X}}{\Delta t} = \frac{\vec{V}_i + \vec{V}_f}{2} \quad \text{if } \vec{a} \text{ is constant}$$

Rearrange that: $\Delta \vec{X} = \frac{1}{2} \cdot \Delta t \cdot (\vec{V}_i + \vec{V}_f)$ if $\vec{a}$ is constant

Substitute values from our graph:

$$\Delta \vec{X} = \frac{1}{2} \cdot 5\,s \cdot ({}^{+}2\,m/s + {}^{+}9.5\,m/s)$$

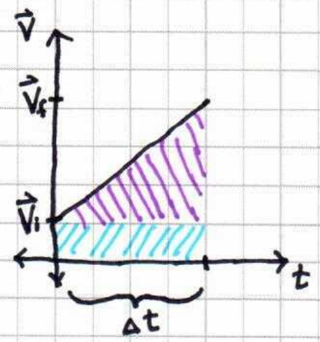
$$\Delta \vec{X} = \frac{1}{2} \cdot 5\,s \cdot {}^{+}11.5\,m/s$$

$$\Delta \vec{X} = {}^{+}28.75\,m$$

Okay, so now we're reasonably certain that our earlier displacement is correct, which implies that our method of getting there is likewise correct,* but let's see if we can clean it up a bit...

* We would actually need to test this in lots more situations, but trust me when I tell you it's correct — you will do plenty more tests soon... :)

We'll sketch our earlier graph one more time:



(a true "sketch" this time)

We decided that the area under the curve was the area of the rectangle plus the area of the triangle:

$$\Delta\vec{x} = A_{total} = A_{\square} + A_{\triangle}$$

$$\Delta\vec{x} = b\cdot h + \frac{1}{2}\cdot b\cdot h$$

$$\Delta\vec{x} = \Delta t\cdot\vec{v}_i + \frac{1}{2}\cdot\Delta t\cdot(\vec{v}_f - \vec{v}_i)$$

Back on page 22, we determined this: $\vec{a} = \frac{\vec{v}_f - \vec{v}_i}{\Delta t}$

Rearranging that just a bit, we get: $\vec{a}\cdot\Delta t = \vec{v}_f - \vec{v}_i$

Now we have this: $\Delta\vec{x} = \vec{v}_i\cdot\Delta t + \frac{1}{2}\cdot(\vec{v}_f - \vec{v}_i)\cdot\Delta t$

and this: $\boxed{\vec{a}\cdot\Delta t} = \vec{v}_f - \vec{v}_i$

Let's substitute this for this and get this:

$$\Delta\vec{x} = \vec{v}_i\cdot\Delta t + \frac{1}{2}\cdot\vec{a}\cdot\Delta t\cdot\Delta t$$ which leads to this:

$$\Delta\vec{x} = \vec{v}_i\cdot\Delta t + \frac{1}{2}\vec{a}\,\Delta t^2$$

Let's make sure that works too. We calculated our acceleration back on page 28 and found $\vec{a} = {}^{+}1.5\ m/s^2$

$$\Delta\vec{x} = \vec{v}_i\,\Delta t + \frac{1}{2}\vec{a}\,\Delta t^2$$

$$\Delta\vec{x} = {}^{+}2\,m/s \cdot 5\,s + \frac{1}{2}\cdot {}^{+}1.5\,m/s^2 \cdot (5\,s)^2$$

$$\Delta\vec{x} = {}^{+}10\,m + {}^{+}18.75\,m$$

$$\Delta\vec{x} = {}^{+}28.75\,m$$

Don't forget to square Δt here – if you forget, you'll be trying to add $^{+}10$ m to $^{+}3.75$ m/s, and that doesn't work – you can't add different units of measure to each other

So this area under the curve stuff seems to work. For our purposes, you can remember it like this:

$$\Delta\vec{x} = \text{area under } \vec{v} \text{ vs } t \text{ curve}$$

In calculus, it looks like this:

$$\Delta\vec{x} = \int \vec{v}\, dt$$

I'll do a calculus page at the end – if you're not ready for that, feel free to ignore it...

On an unrelated note, if an object is traveling at a velocity of + 8 m/s and accelerating at − 3 m/s², what is its location? In other words, where is the object? What is its position (x)?

Hopefully you concluded that there's no way to know the objects' position without more information.

Having a $\vec{v}$ vs t graph and/or knowing an object's velocity and rate of acceleration can only give you its change in position ($\Delta\vec{x}$) – it tells you nothing about where it started (x_i) nor where it ends up (x_f). You can see this in our latest equation: $\Delta\vec{x} = \vec{v}_i \cdot \Delta t + \frac{1}{2}\vec{a}\Delta t^2$

Since $\Delta\vec{x} = x_f - x_i$, we can rewrite that as:

$$x_f - x_i = \frac{1}{2}\vec{a}\cdot\Delta t^2 + \vec{v}_i\cdot\Delta t$$ and then rearrange:

$$x_f = \frac{1}{2}\vec{a}\cdot\Delta t^2 + \vec{v}_i\cdot\Delta t + x_i$$

★ You would have to know either x_i or x_f to determine the other one...

Notice that the above equation, which happens to be a general equation for motion in one dimension, can be simplified in some circumstances:

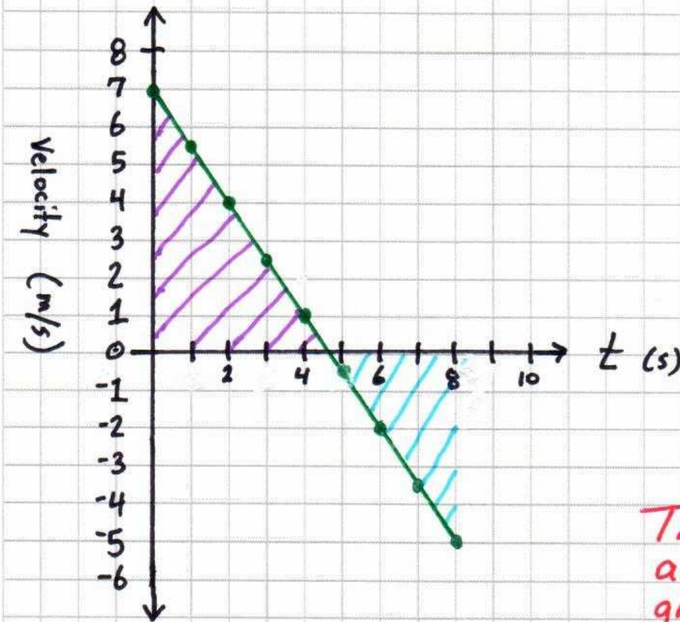
If $\vec{a} = \varnothing$: $x_f = \underbrace{\cancel{\frac{1}{2}\vec{a}\Delta t^2}}_{}^{\vec{a}=\varnothing} + \vec{v}_i\Delta t + x_i \rightarrow x_f = \vec{v}\cdot\Delta t + x_i$

which can be rearranged to: $\Delta\vec{x} = \vec{v}\cdot\Delta t$

If $\vec{v}_i = \varnothing$: $x_f = \frac{1}{2}\vec{a}\Delta t^2 + \overset{\vec{v}_i=\varnothing}{\cancel{\vec{v}_i\Delta t}} + x_i \rightarrow x_f = \frac{1}{2}\vec{a}\Delta t^2 + x_i$

which can be rearranged to: $\Delta\vec{x} = \frac{1}{2}\vec{a}\Delta t^2$

As a practice exercise, let's look back at our graph of the fan cart's motion on page 20 and use what we're learned here to determine what we can:



Approach #1:

$\Delta\vec{X}$ = area under curve

$\frac{1}{2} \cdot \sim 4.7\,s \cdot {}^{+}7\ m/s \cong {}^{+}16.5\ m$

$\frac{1}{2} \cdot \sim 3.3\,s \cdot {}^{-}5\ m/s \cong {}^{-}8.25\ m$

$\Delta\vec{X} = {}^{+}16.5\ m + {}^{-}8.25\ m$

$\Delta\vec{X} = +8.25\ m$

This is an approximation, because we are unable to say exactly where the graph crosses the time axis. This should get us pretty close to the actual value of $\Delta\vec{X}$ though...

Approach #2:

$\Delta\vec{X} = \vec{V}_{avg} \cdot \Delta t$

$\vec{V}_{avg} = \frac{\vec{V}_i + \vec{V}_f}{2}$ since $\vec{a}$ is constant

Therefore:

$\Delta\vec{X} = \frac{1}{2} \cdot (\vec{V}_i + \vec{V}_f) \cdot \Delta t$

$\Delta\vec{X} = \frac{1}{2} \cdot ({}^{+}7\ m/s + {}^{-}5\ m/s) \cdot 8\,s$

$\Delta\vec{X} = \frac{1}{2} \cdot {}^{+}2\ m/s \cdot 8\,s$

$\Delta\vec{X} = {}^{+}8\ m$

Pretty close to ${}^{+}8.25\ m$ ...

Approach #3:

$\Delta\vec{X} = \vec{V}_i \Delta t + \frac{1}{2}\vec{a}\Delta t^2$ if constant $\vec{a}$ (and it is)

but we need $\vec{a}$:

$\vec{a} = \frac{\vec{V}_f - \vec{V}_i}{\Delta t} = \frac{{}^{-}5\ m/s - {}^{+}7\ m/s}{8\,s} = \frac{{}^{-}12\ m/s}{8\,s} = {}^{-}1.5\ m/s^2$

$\Delta\vec{X} = {}^{+}7\ m/s \cdot 8\,s + \frac{1}{2} \cdot {}^{-}1.5\ m/s^2 \cdot (8\,s)^2$

$\Delta\vec{X} = {}^{+}56\ m + {}^{-}48\ m$

$\Delta\vec{X} = {}^{+}8\ m$

Again, pretty close to ${}^{+}8.25\ m$

Everything so far is pretty useful in basic high school physics, but those pesky textbook authors love to give practice problems that don't have a time interval specified. This can cause issues getting solutions, but have no fear: algebra is here!

We developed the following three equations along the way to get here:

if constant $\vec{a}$

$$\Delta\vec{x} = \vec{V}_{avg} \cdot \Delta t \quad ; \quad \vec{V}_{avg} = \frac{\vec{V}_i + \vec{V}_f}{2} \quad ; \quad \vec{a} = \frac{\vec{V}_f - \vec{V}_i}{\Delta t}$$

The first two are easy enough to combine — we've already done that a few times:

$$\Delta\vec{x} = \frac{\vec{V}_i + \vec{V}_f}{2} \cdot \Delta t$$

Now rearrange the acceleration equation: $\Delta t = \dfrac{\vec{V}_f - \vec{V}_i}{\vec{a}}$

Plug in that for Δt in the above equation:

$$\Delta\vec{x} = \frac{\vec{V}_i + \vec{V}_f}{2} \cdot \frac{\vec{V}_f - \vec{V}_i}{\vec{a}}$$

Multiply both sides by $2\vec{a}$: $2\vec{a}\,\Delta x = (\vec{V}_i + \vec{V}_f)(\vec{V}_f - \vec{V}_i)$

Multiply the binomials together: $2\vec{a}\,\Delta\vec{x} = V_f^{\,2} - V_i^{\,2}$ {behold the timeless equation}

Note that there are no $\overrightarrow{\text{vector}}$ signs on V_f and V_i here — since they're squared, the + or − sign becomes meaningless (either one is positive when squared)

We've talked a lot about objects moving horizontally (i.e. along an x-axis) in one dimension, so it seems reasonable for us to consider the case of an object moving vertically (i.e. along a y-axis) in one dimension.

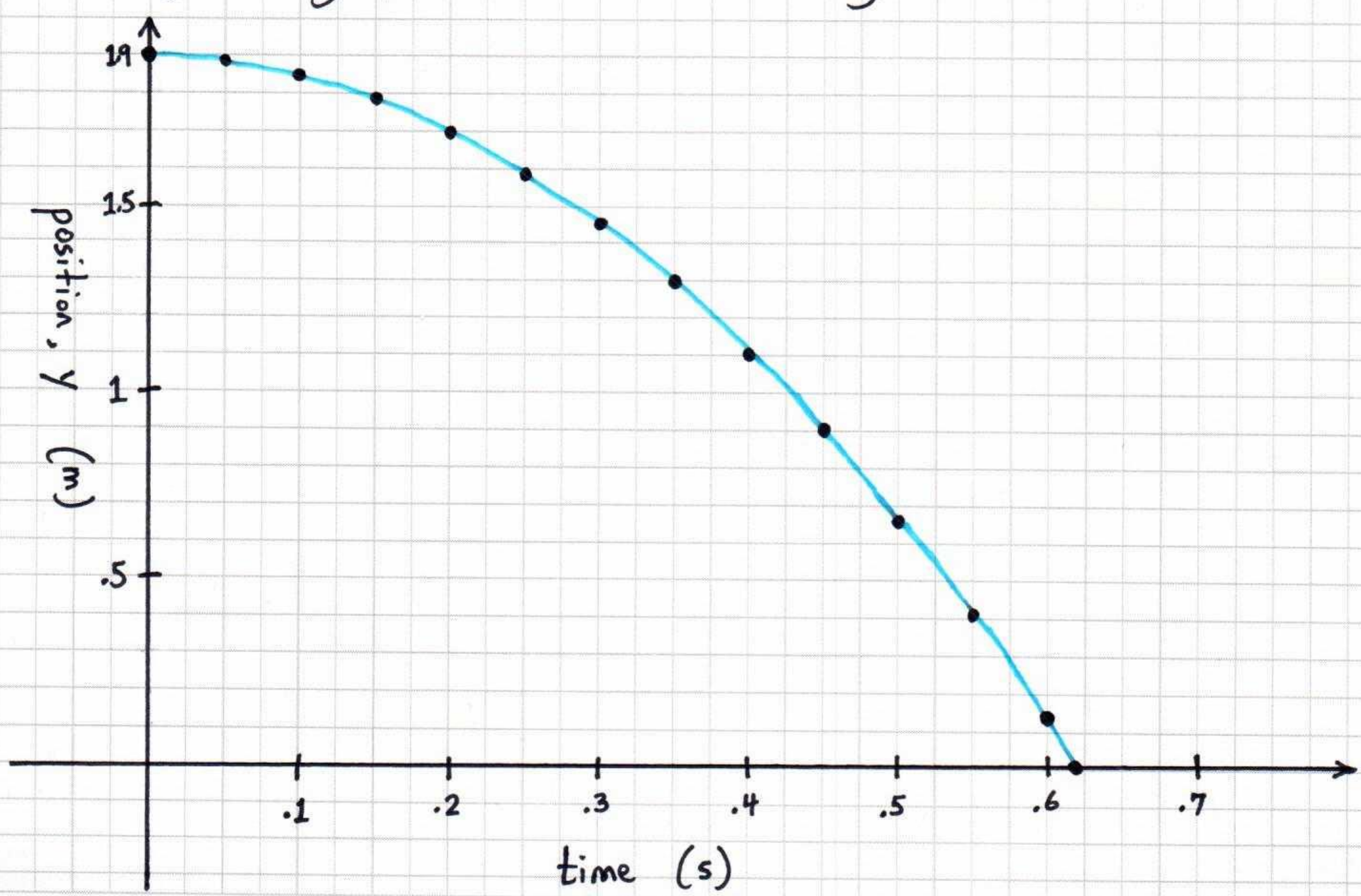
On Earth, an object moving vertically (up or down) without interference or help from other objects — i.e. the object is thrown or dropped rather that being carried or pulled by a rope — is under the influence of gravity. We all know that, of course, but what is gravity? More specifically, what does gravity do to the object?

In Aristotle's time, everyone knew that heavier objects would fall faster than lighter objects. As it turns out, that was wrong. In the absence of air resistance, heavy objects don't fall any faster than light objects. The "why" behind that is left as an open question for later in the course, but confirmation <u>that</u> it occurs this way is easy enough.

We know from experience that falling objects speed up as they fall, i.e. their velocity increases in the "down" direction.

Since a falling object's velocity increases in the "down" direction as it falls, and a change in velocity necessarily implies acceleration, we can reasonably say that gravity causes objects to accelerate downward. That's a good qualitative conclusion, but it would be nice to get something more quantitative, i.e. some number — exactly what is the rate of acceleration due to gravity?

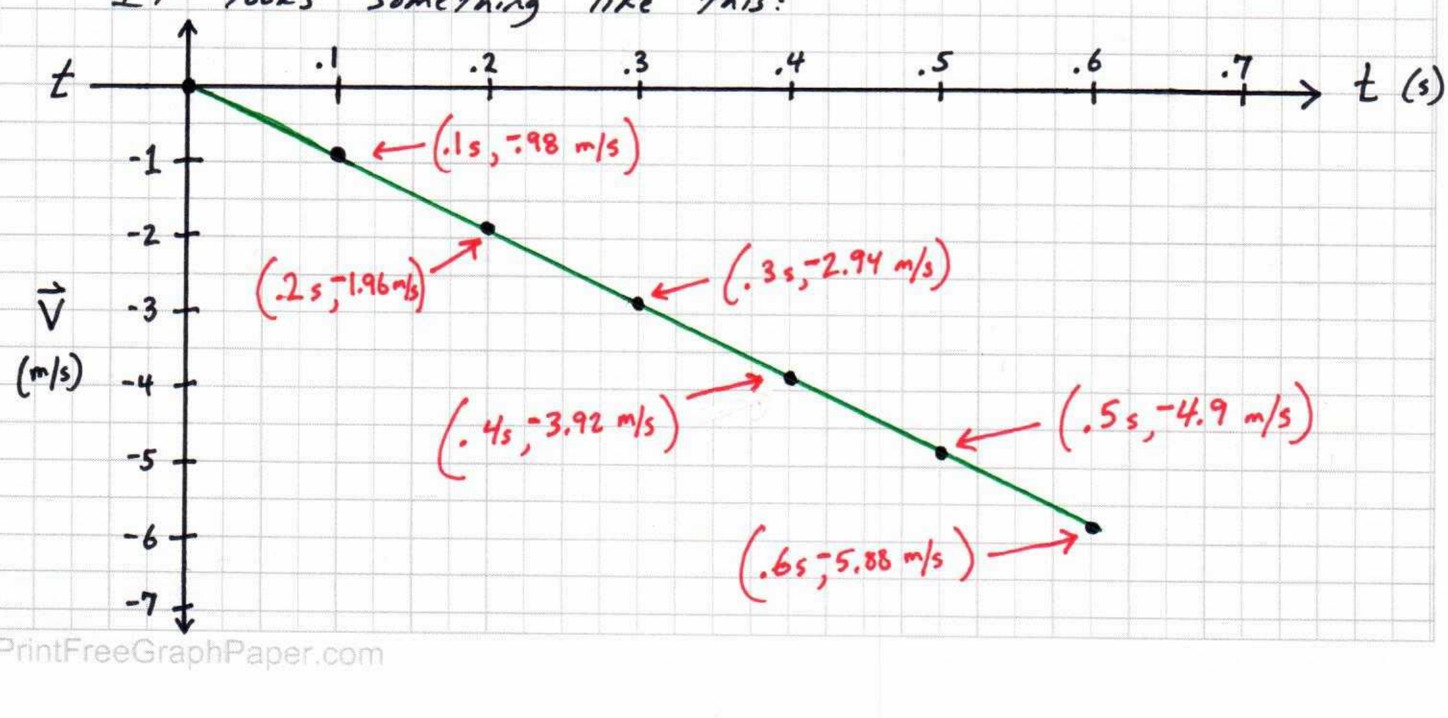
One way to find out is to use a motion sensor and a textbook. We'll attach a motion sensor to the ceiling, hold a textbook underneath it, and drop the book. We'll get a graph that looks something like this:



As you can see on the graph, the book started 1.9 m above the floor and landed on the floor about .62 seconds later. We can see that the book is speeding up as it falls: from 0 to .1 s, it falls about .05 m, whereas from .5 to .6 s (an equal amount of time), it falls about .5 m.

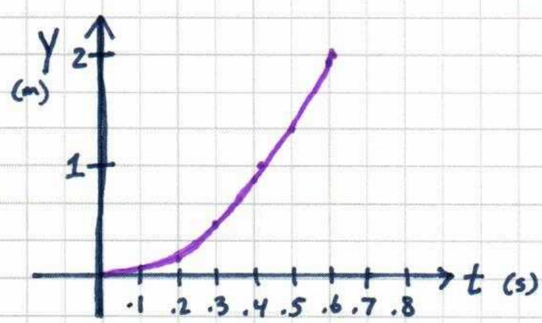
That confirms our qualitative conclusion from earlier — gravity definitely accelerates things downward - but we still don't have a number for it.

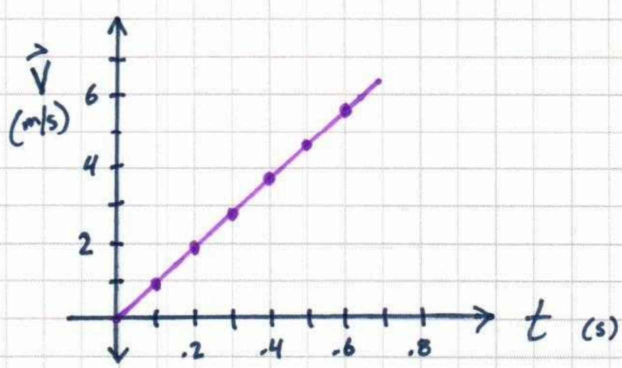
We could spend a lot of time finding the slope of the graph at various points (slope of position vs. time is velocity, remember) and then compare the changes in velocities of the time intervals, but the software we used to collect the data and produce the graph can give us a velocity vs time graph of the same motion. It looks something like this:



Notice that the velocity goes from zero (that's sensible – you were holding still before you dropped it) to −.98 m/s to −1.96 m/s and so on, getting more and more negative as time passes. This is consistent with our expectations: we saw on our position vs time graph that the book started at +1.9 m and fell to 0 m, so its change in position ($\Delta \vec{y}$) was negative. That requires motion in the negative direction, hence the negative velocities.

★ Note that there's no requirement for down to be negative; we could just as easily have made down the positive direction, and our graphs would have resembled these sketches:





★ Notice how that y vs t graph almost looks linear after .3 s? That's the danger of scaling your axes too small – you lose too much detail...

Anyway, let's go back to the first two graphs of the book...

Our goal was to find the rate of acceleration due to gravity. So far, we've concluded that it's always downward, and now thanks to our graph of $\vec{V}$ vs t, we know the rate is constant — our $\vec{V}$ vs t graph is linear, so the change in $\vec{V}$ relative to change in t (the slope) is constant. Hmm... slope...

We learned earlier that slope of a $\vec{V}$ vs t graph is acceleration:

$$\vec{a} = \frac{\Delta\vec{V}}{\Delta t} = \frac{\vec{V}_f - \vec{V}_i}{t_f - t_i}$$

so let's pick a couple of points on our line:

At $t = .2\,s$, $\vec{V} = -1.96\,m/s$; At $t = .5\,s$, $\vec{V} = -4.9\,m/s$

$$\vec{a} = \text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta\vec{V}}{\Delta t} = \frac{\vec{V}_f - \vec{V}_i}{t_f - t_i} = \frac{-4.9\,m/s - {}^{-}1.96\,m/s}{.5\,s - .2\,s} =$$

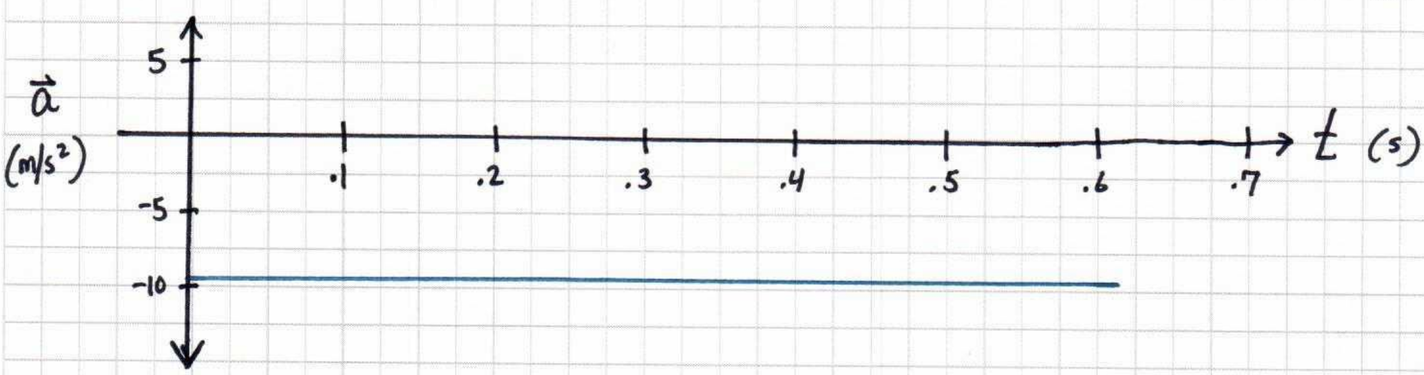
$$= \frac{-2.94\,m/s}{.3\,s} = -9.8\,m/s^2$$

★ Objects in freefall experience acceleration of $9.8\,m/s^2$ downward — their change in velocity is $9.8\,m/s$ downward during every second

★ $g = 9.8\,m/s^2$

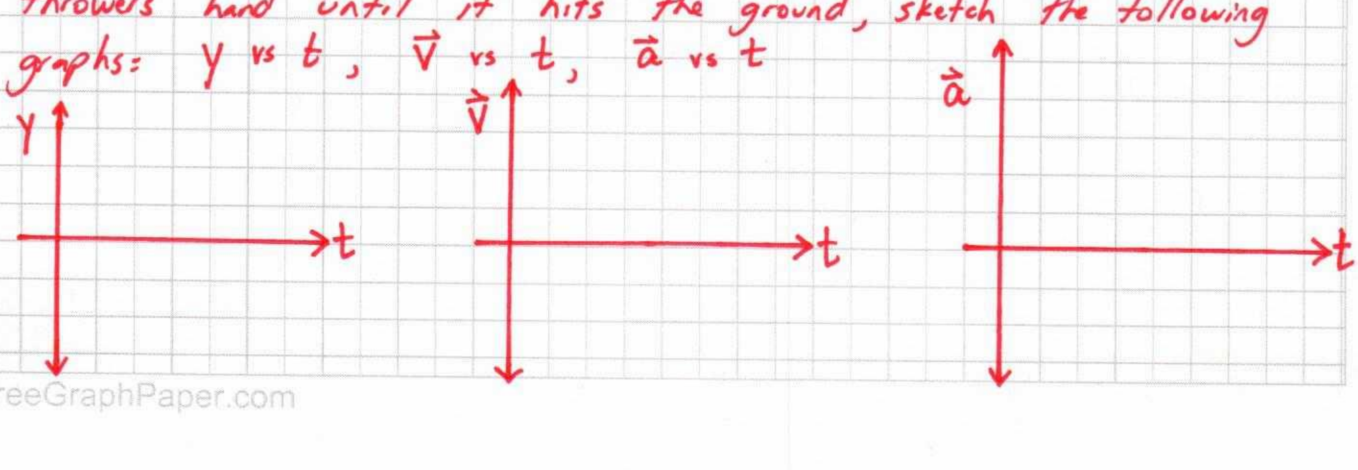
The constant known as "g" is itself a simple quantity — you have to assign direction to it as appropriate...

Ideally, you're now curious about what the book's $\vec{a}$ vs t graph would look like. If so, you're in luck. We just determined that $\vec{a}$ due to gravity is constant (absent air resistance). In other words, at every point in time while the book was falling, its acceleration was 9.8 m/s^2 toward the ground (-9.8 m/s^2 in this case since we declared down to be the negative direction). This, therefore, is an easy graph:



$\vec{a}$ is -9.8 m/s^2 the entire time, as expected.

Check your understanding — imagine a ball thrown straight up into the air. For the time period between it leaving the thrower's hand until it hits the ground, sketch the following graphs: y vs t, $\vec{v}$ vs t, $\vec{a}$ vs t



Calculus as it pertains to 1D Motion

Assuming constant $\vec{a}$, motion in 1D can be described with:

$$X = \frac{1}{2}\vec{a}\,\Delta t^2 + \vec{V}_i\,\Delta t + X_i \qquad \text{OR} \quad \Delta\vec{X} = \frac{1}{2}\vec{a}\,\Delta t^2 + \vec{V}_i\,\Delta t$$

$$\frac{dx}{dt} = \vec{V} = \vec{a}\,\Delta t + \vec{V}_i \qquad \text{OR} \quad \Delta\vec{V} = \vec{a}\,\Delta t \quad \text{OR} \quad \vec{a} = \frac{\Delta\vec{V}}{\Delta t}$$

$$\frac{d\vec{v}}{dt} = \vec{a} = \vec{a} \qquad \text{OR} \quad \vec{a}$$

★ sorry, I couldn't resist ☺

$$\Delta\vec{X} = \text{area under } \vec{V} \text{ vs } t \text{ curve} = \int \vec{v}\, dt = \int \left(\vec{a}\,\Delta t + \vec{V}_i\right) dt =$$

$$= \frac{1}{2}\vec{a}\,\Delta t^2 + \vec{V}_i\,\Delta t = \Delta\vec{X}$$

(see line 1)

★ In Calculus class, you'll be reminded to always add "+ C" when integrating. We are NOT violating that rule: technically, what we get is this: $\frac{1}{2}\vec{a}\,\Delta t^2 + \vec{V}_i\,\Delta t + C$ but $C = X_i$, so we subtract it from each side of the equation, which leaves "$X - X_i$" on the left side, and that's $\Delta\vec{X}$ ☺

$$\Delta\vec{V} = \text{area under } \vec{a} \text{ vs } t \text{ curve} = \int \vec{a}\, dt = \vec{a}\Delta t + C$$

but $C = \vec{V}_i$, so: $\int \vec{a}\, dt = \vec{V} = \vec{a}\,\Delta t + \vec{V}_i \rightarrow \Delta\vec{V} = \vec{a}\,\Delta t$

That pretty much covers everything but the timeless equation. We can make a little time for that on the next page:

How does $\Delta\vec{V}$ relate to $\Delta\vec{x}$? Hmm...

$\frac{dv}{dx} = ?$ Multiply it by 1 $\left(\frac{dt}{dt}\right)$: $\frac{dv}{dx} \cdot \frac{dt}{dt}$

$$\frac{dv}{dx} \cdot \frac{dt}{dt} = \frac{dv}{dt} \cdot \frac{dt}{dx}$$ (Rearranged the parts)

$\frac{dv}{dt} = \vec{a}$ and $\frac{dx}{dt} = \vec{V}$, so $\frac{dt}{dx} = \frac{1}{\vec{V}}$:

$\frac{dv}{dt} \cdot \frac{dt}{dx} = \vec{a} \cdot \frac{1}{\vec{V}}$ We started with $\frac{dv}{dx} = ?$

So... $\frac{dv}{dx} = \frac{\vec{a}}{\vec{V}} \xrightarrow{\text{rearranged}} \vec{V}\, dv = \vec{a}\, dx$

Integrate both sides: $\int \vec{V}\, dv = \int \vec{a}\, dx \longrightarrow$

$$\frac{1}{2}\left(V^2 - V_i^2\right) = a\left(x - x_i\right) \xrightarrow{\text{simple algebra}}$$

$$\vec{V}_f^{\,2} - \vec{V}_i^{\,2} = 2\vec{a}\,\Delta\vec{x}$$